MATH20132 Calculus of Several Variable. 2020-21

Solutions to Problems 3: The Directional Derivative

1 Define the functions

i. $f : \mathbb{R}^2 \to \mathbb{R}$, $\mathbf{x} \mapsto x(x+y)$ and

ii. $g : \mathbb{R}^2 \to \mathbb{R}$, $\mathbf{x} \mapsto y(x-y)$.

Find the directional derivatives of f and g at $\mathbf{a} = (1,2)^T$ in the direction $\mathbf{v} = (2,-1)^T/\sqrt{5}$.

Solution First note that

$$\mathbf{a} + t\mathbf{v} = \begin{pmatrix} 1 + 2t/\sqrt{5} \\ 2 - t/\sqrt{5} \end{pmatrix}.$$

So

$$f(\mathbf{a} + t\mathbf{v}) = \left(1 + \frac{2t}{\sqrt{5}}\right)\left(3 + \frac{t}{\sqrt{5}}\right) = 3 + \frac{7}{\sqrt{5}}t + \frac{2}{5}t^2.$$

Thus $f(\mathbf{a}) = 3$ and

$$\frac{f(\mathbf{a} + t\mathbf{v}) - f(\mathbf{a})}{t} = \frac{7}{\sqrt{5}} + \frac{2}{5}t \to \frac{7}{\sqrt{5}}$$

as $t \to 0$. Since the limit exists the directional derivative exists and satisfies $d_{\mathbf{v}}f(\mathbf{a}) = 7/\sqrt{5}$.

For g we have

$$g(\mathbf{a} + t\mathbf{v}) = \left(2 - \frac{t}{\sqrt{5}}\right)\left(-1 + \frac{3t}{\sqrt{5}}\right) = -2 + \frac{7}{\sqrt{5}}t - \frac{3}{5}t^2.$$

Thus $g(\mathbf{a}) = -2$ and

$$\frac{g(\mathbf{a} + t\mathbf{v}) - g(\mathbf{a})}{t} = \frac{7}{\sqrt{5}} - \frac{3}{5}t \to \frac{7}{\sqrt{5}},$$

as $t \to 0$. Since the limit exists the directional derivative exists and satisfies $d_{\mathbf{v}}g(\mathbf{a}) = 7/\sqrt{5}$.

2. Find the directional derivative of $f : \mathbb{R}^2 \to \mathbb{R}$, $\mathbf{x} \to x^2y$ at $\mathbf{a} = (2,1)^T$ in the direction of the unit vector $\mathbf{v} = (1,-1)^T/\sqrt{2}$.

Solution First note that

$$\mathbf{a} + t\mathbf{v} = \begin{pmatrix} 2 + t/\sqrt{2} \\ 1 - t/\sqrt{2} \end{pmatrix},$$

so

$$f(\mathbf{a} + t\mathbf{v}) = \left(2 + \frac{t}{\sqrt{2}}\right)^2 \left(1 - \frac{t}{\sqrt{2}}\right) = 4 - \frac{3}{2}t^2 - \frac{1}{2\sqrt{2}}t^3.$$

This leads to the existence of the directional derivative and it's value $d_{\mathbf{v}} f(\mathbf{a}) = 0$.

3. Define the function $h : \mathbb{R}^3 \to \mathbb{R}$, by $\mathbf{x} \to xy + yz + xz$. By verifying the definition, find the directional derivative of h at $\mathbf{a} = (1, 2, 3)^T$ in the direction of the unit vector $\mathbf{v} = (3, 2, 1)^T / \sqrt{14}$.

Solution First note that

$$\mathbf{a} + \mathbf{v}t = \begin{pmatrix} 1 + 3t/\sqrt{14} \\ 2 + 2t/\sqrt{14} \\ 3 + t/\sqrt{14} \end{pmatrix}.$$

So

$$\begin{aligned} h(\mathbf{a} + \mathbf{v}t) &= \left(1 + 3\frac{t}{\sqrt{14}}\right)\left(2 + 2\frac{t}{\sqrt{14}}\right) + \left(2 + 2\frac{t}{\sqrt{14}}\right)\left(3 + \frac{t}{\sqrt{14}}\right) \\ &\quad + \left(1 + 3\frac{t}{\sqrt{14}}\right)\left(3 + \frac{t}{\sqrt{14}}\right) \\ &= 2 + 8\frac{t}{\sqrt{14}} + 6\frac{t^2}{14} + 6 + 8\frac{t}{\sqrt{14}} + 2\frac{t^2}{14} + 3 + 10\frac{t}{\sqrt{14}} + 3\frac{t^2}{14} \\ &= 11 + 26\frac{t}{\sqrt{14}} + 11\frac{t^2}{14}. \end{aligned}$$

Then

$$\begin{aligned} \frac{h(\mathbf{a} + \mathbf{v}t) - h(\mathbf{a})}{t} &= \frac{1}{t}\left(26\frac{t}{\sqrt{14}} + 11\frac{t^2}{14}\right) = 26\frac{1}{\sqrt{14}} + 11\frac{t}{14} \\ &\to \frac{26}{\sqrt{14}}, \end{aligned}$$

as $t \to 0$. Since the limit exists the directional derivative exists and satisfies $d_{\mathbf{v}} h(\mathbf{a}) = 26/\sqrt{14}$.

4. Define the function $f : \mathbb{R}^3 \to \mathbb{R}$, by $\mathbf{x} \to xy^2z$. By verifying the definition, find the directional derivative of $\mathbf{f}$ at $\mathbf{a} = (1, 3, -2)^T$ in the direction of the unit vector $\mathbf{v} = (-1, 1, -2)^T/\sqrt{6}$.

Solution Firstly,

$$\mathbf{a} + t\mathbf{v} = \begin{pmatrix} 1 - t/\sqrt{6} \\ 3 + t/\sqrt{6} \\ -2 - 2t/\sqrt{6} \end{pmatrix}.$$

Then

$$\begin{aligned} f(\mathbf{a} + t\mathbf{v}) &= \left(1 - \frac{t}{\sqrt{6}}\right)\left(3 + \frac{t}{\sqrt{6}}\right)^2\left(-2 - 2\frac{t}{\sqrt{6}}\right) \\ &= -18 - 2t\sqrt{6} + \frac{8}{3}t^2 + \frac{1}{3}t^3\sqrt{6} + \frac{1}{18}t^4. \end{aligned}$$

You do not need all this detail, instead write it as $-18 - 2t\sqrt{6} + O(t^2)$, where the $O(t^2)$ notation represents the sum of all terms with t^2 or higher powers.

This will lead us to $d_{\mathbf{v}}f(\mathbf{a}) = -2\sqrt{6}$.

5. Define the function $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$\mathbf{x} \to \begin{pmatrix} xy \\ yz \end{pmatrix},$$

where $\mathbf{x} = (x, y, z)^T$. By verifying the definition, find the directional derivative of $\mathbf{f}$ at $\mathbf{a} = (1, 3, -2)^T$ in the direction of the unit vector $\mathbf{v} = (-1, 1, -2)^T/\sqrt{6}$.

Do **not** look at the component functions separately.

Solution Consider, for $t \neq 0$,

$$\begin{aligned} \frac{\mathbf{f}(\mathbf{a} + t\mathbf{v}) - \mathbf{f}(\mathbf{a})}{t} &= \frac{1}{t}\left\{\begin{pmatrix} (1 - t/\sqrt{6})\,(3 + t/\sqrt{6}) \\ (3 + t/\sqrt{6})\,(-2 - 2t/\sqrt{6}) \end{pmatrix} - \begin{pmatrix} 3 \\ -6 \end{pmatrix}\right\} \\ &= \frac{1}{t}\begin{pmatrix} -2t/\sqrt{6} - t^2/6 \\ -8t/\sqrt{6} - 2t^2/6 \end{pmatrix} = \begin{pmatrix} -2/\sqrt{6} - t/6 \\ -8/\sqrt{6} - 2t/6 \end{pmatrix} \\ &\to \begin{pmatrix} -2/\sqrt{6} \\ -8/\sqrt{6} \end{pmatrix} \qquad \text{as } t \to 0, \\ &= -\sqrt{\frac{2}{3}}\begin{pmatrix} 1 \\ 4 \end{pmatrix}. \end{aligned}$$

Since the limit exists the directional derivative exists and satisfies

$$d_{\mathbf{v}}\mathbf{f}\left(\mathbf{a}\right) = -\sqrt{\frac{2}{3}}\begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

6 Define the function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x\left(x+y\right) \\ y\left(x-y\right) \end{pmatrix}.$$

Find the directional derivative of $\mathbf{f}$ at $\mathbf{a} = (1,2)^T$ in the direction $\mathbf{v} = (2,-1)^T/\sqrt{5}$.

Hint Notice the difference in wording between this question and the previous one; here I do not ask you to verify the definition.

Solution Use the result that the directional derivative of a vector-valued function exists iff the directional derivatives of it's component functions exist and satisfy $d_{\mathbf{v}}\mathbf{f}\left(\mathbf{a}\right)^i = d_{\mathbf{v}}f^i\left(\mathbf{a}\right)$. In this case the component functions have been seen in Question 1, where their directional derivatives were shown to exist and thus $d_{\mathbf{v}}\mathbf{f}\left(\mathbf{a}\right)$ exists. Further,

$$d_{\mathbf{v}}\mathbf{f}(\mathbf{a}) = \begin{pmatrix} d_{\mathbf{v}}f^1(\mathbf{a}) \\ d_{\mathbf{v}}f^2(\mathbf{a}) \end{pmatrix} = \begin{pmatrix} 7/\sqrt{5} \\ 7/\sqrt{5} \end{pmatrix}.$$

7 Define the function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} xy^2 \\ x^2y \end{pmatrix}.$$

Find the directional derivative of $\mathbf{f}$ at $\mathbf{a} = (2,1)^T$ in the direction $\mathbf{v} = (1,-1)^T/\sqrt{5}$.

Solution $d_{\mathbf{v}}\mathbf{f}(\mathbf{a})$ exists iff $d_{\mathbf{v}}f^1(\mathbf{a})$ and $d_{\mathbf{v}}f^2(\mathbf{a})$. Here $f^1(\mathbf{x}) = xy^2$ was an example in lectures where we found $d_{\mathbf{v}}f^1(\mathbf{a}) = -3/\sqrt{2}$. And $f^2(\mathbf{x}) = x^2y$ was the subject of Question 2 above where we found that $d_{\mathbf{v}}f^2(\mathbf{a}) = 0$. Hence

$$d_{\mathbf{v}}\mathbf{f}(\mathbf{a}) = \begin{pmatrix} d_{\mathbf{v}}f^1(\mathbf{a}) \\ d_{\mathbf{v}}f^2(\mathbf{a}) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

8.

i. Let $\mathbf{c} \in \mathbb{R}^n$ be fixed. Let $f : \mathbb{R}^n \to \mathbb{R}, \mathbf{x} \mapsto \mathbf{c} \bullet \mathbf{x}$. Show that

$$d_{\mathbf{v}} f(\mathbf{a}) = f(\mathbf{v})$$

for all $\mathbf{a}, \mathbf{v} \in \mathbb{R}^n$.

ii. Let $M \in M_{m,n}(\mathbb{R})$ and $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m, \mathbf{x} \mapsto M\mathbf{x}$. Show that

$$d_{\mathbf{v}} \mathbf{f}(\mathbf{a}) = \mathbf{f}(\mathbf{v})$$

for all $\mathbf{a}, \mathbf{v} \in \mathbb{R}^n$.

iii. Can you generalise these results? I.e. of what type of function are $\mathbf{x} \mapsto \mathbf{c} \bullet \mathbf{x}$ and $\mathbf{x} \mapsto M\mathbf{x}$ examples?

Solution i. Let $\mathbf{a}, \mathbf{v} \in \mathbb{R}^n$ be given. Consider

$$\begin{aligned} \frac{\mathbf{f}(\mathbf{a} + t\mathbf{v}) - \mathbf{f}(\mathbf{a})}{t} &= \frac{1}{t}(\mathbf{c} \bullet (\mathbf{a} + t\mathbf{v}) - \mathbf{c} \bullet \mathbf{a}) = \frac{1}{t}(\mathbf{c} \bullet \mathbf{a} + t\mathbf{c} \bullet \mathbf{v} - \mathbf{c} \bullet \mathbf{a}) \\ &\qquad \text{since the scalar product is distributive} \\ &= \mathbf{c} \bullet \mathbf{v} = \mathbf{f}(\mathbf{v}), \end{aligned}$$

for all $t \neq 0$. Hence

$$\lim_{t \to 0} \frac{\mathbf{f}(\mathbf{a} + t\mathbf{v}) - \mathbf{f}(\mathbf{a})}{t} = \mathbf{f}(\mathbf{v}).$$

That the limit exists means that the directional derivative exists. That the limit is $\mathbf{f}(\mathbf{v})$ means that $d_{\mathbf{v}}\mathbf{f}(\mathbf{a}) = \mathbf{f}(\mathbf{v})$.

ii. Let $\mathbf{a}, \mathbf{v} \in \mathbb{R}^n$ be given. Consider

$$\begin{aligned} \frac{\mathbf{f}(\mathbf{a} + t\mathbf{v}) - \mathbf{f}(\mathbf{a})}{t} &= \frac{1}{t}(M(\mathbf{a} + t\mathbf{v}) - M\mathbf{a}) = \frac{1}{t}(M\mathbf{a} + tM\mathbf{v} - M\mathbf{a}) \\ &\qquad \text{since matrix multiplication is distributive} \\ &= M\mathbf{v} = \mathbf{f}(\mathbf{v}), \end{aligned}$$

for all $t \neq 0$. Hence

$$\lim_{t \to 0} \frac{\mathbf{f}(\mathbf{a} + t\mathbf{v}) - \mathbf{f}(\mathbf{a})}{t} = \mathbf{f}(\mathbf{v}).$$

That the limit exists means that the directional derivative exists. That the limit is $\mathbf{f}(\mathbf{v})$ means that $d_{\mathbf{v}}\mathbf{f}(\mathbf{a}) = \mathbf{f}(\mathbf{v})$.

iii. Both $\mathbf{x} \mapsto \mathbf{c} \bullet \mathbf{x}$ and $\mathbf{x} \mapsto M\mathbf{x}$ are examples of linear functions. Let $\mathbf{L} : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function. Let $\mathbf{a}, \mathbf{v} \in \mathbb{R}^n$ be given. Consider

$$\begin{aligned} \frac{\mathbf{L}(\mathbf{a}+t\mathbf{v}) - \mathbf{L}(\mathbf{a})}{t} &= \frac{\mathbf{L}(\mathbf{a}) + t\mathbf{L}(\mathbf{v}) - \mathbf{L}(\mathbf{a})}{t} \\ &\qquad \text{by the linearity of } \mathbf{L} \\ &= \mathbf{L}(\mathbf{v}), \end{aligned}$$

for all $t \neq 0$. Hence

$$\lim_{t\to 0} \frac{\mathbf{L}(\mathbf{a}+t\mathbf{v}) - \mathbf{L}(\mathbf{a})}{t} = \mathbf{L}(\mathbf{v}).$$

That the limit exists means that the directional derivative exists. That the limit is $\mathbf{L}(\mathbf{v})$ means that $d_{\mathbf{v}}\mathbf{L}(\mathbf{a}) = \mathbf{L}(\mathbf{v})$.

9. Assume for the scalar-valued function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ the directional derivative $d_{\mathbf{v}}f(\mathbf{a})$ exists for some $\mathbf{a}, \mathbf{v} \in \mathbb{R}^n$. Prove that

$$\lim_{t\to 0} f(\mathbf{a}+t\mathbf{v}) = f(\mathbf{a}).$$

This is yet another example of the principle that if a function is differentiable at a point then it is continuous at that point. There are no new ideas in the proof, look back at previous proofs of differentiable implies continuous.

Solution This is a proof you should recognise from earlier analysis courses. Consider

$$\lim_{t\to 0}\left(f(\mathbf{a}+\mathbf{v}t) - f(\mathbf{a})\right) = \lim_{t\to 0} \frac{f(\mathbf{a}+\mathbf{v}t) - f(\mathbf{a})}{t} t = \lim_{t\to 0} \frac{f(\mathbf{a}+\mathbf{v}t) - f(\mathbf{a})}{t} \lim_{t\to 0} t,$$

using the Product Rule for limits, allowable only if the two limits exist. The second limit is 0, the first is $d_{\mathbf{v}}f(\mathbf{a})$ which exists by assumption. Hence

$$\lim_{t\to 0}\left(f(\mathbf{a}+\mathbf{v}t) - f(\mathbf{a})\right) = d_{\mathbf{v}}f(\mathbf{a}) \times 0 = 0,$$

which gives required result.

10. Define the function $f : \mathbb{R}^n \to \mathbb{R}$ by $\mathbf{x} \to |\mathbf{x}|$.

i. Prove that f is continuous in any direction at the origin.

ii. Show that in no direction through the origin does f have a directional derivative.

This example illustrates the fact that

continuous in a direction $\not\Longrightarrow$ differentiable in that direction.

Solution i. Let $\mathbf{v}$, a unit vector, be given. Then

$$f(\mathbf{0}+t\mathbf{v}) = |t\mathbf{v}| = |t|\,|\mathbf{v}| \underset{t\to 0}{\to} 0 = f(\mathbf{0}).$$

Hence f is continuous at $\mathbf{0}$ in the direction $\mathbf{v}$. Yet $\mathbf{v}$ was arbitrary, so f is continuous in any direction at the origin.

ii. Let $\mathbf{v}$, a unit vector, be given. Then

$$\frac{f(\mathbf{0}+t\mathbf{v}) - f(\mathbf{0})}{t} = \frac{|t|\,|\mathbf{v}|}{t}.$$

It is well-known that $\lim_{t\to 0} |t|/t$ does not exist; the right hand and left hand limits are different. Hence

$$\lim_{t\to 0} \frac{f(\mathbf{0}+t\mathbf{v}) - f(\mathbf{0})}{t}$$

does not exist, i.e. f has no directional derivative at 0 in the direction of $\mathbf{v}$. Yet $\mathbf{v}$ was arbitrary, so in no direction through the origin does f have a directional derivative.

11. Assume $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$, $\mathbf{a} \in U$ and we have a unit vector $\mathbf{v} \in \mathbb{R}^n$. Prove that if the directional derivative $d_{\mathbf{v}}f(\mathbf{a})$ exists then so does the directional derivative $d_{-\mathbf{v}}f(\mathbf{a})$ and that it satisfies $d_{-\mathbf{v}}f(\mathbf{a}) = -d_{\mathbf{v}}f(\mathbf{a})$.

Solution Consider the definition of $d_{-\mathbf{v}}f(\mathbf{a})$,

$$\begin{aligned}
\lim_{t\to 0} \frac{f(\mathbf{a}+(-\mathbf{v})t) - f(\mathbf{a})}{t} &= \lim_{t\to 0} \frac{f(\mathbf{a}-\mathbf{v}t) - f(\mathbf{a})}{t} \\
&= \lim_{s\to 0} \frac{f(\mathbf{a}+\mathbf{v}s) - f(\mathbf{a})}{-s} \quad \text{putting } s=-t \\
&= -d_{\mathbf{v}}f(\mathbf{a}).
\end{aligned}$$

That the limit exists means that $d_{-\mathbf{v}}f(\mathbf{a})$ exists and further satisfies $d_{-\mathbf{v}}f(\mathbf{a}) = -d_{\mathbf{v}}f(\mathbf{a})$.

12. Using the definition of directional derivative calculate $d_1(x^2y)$ and $d_2(x^2y)$. Hence verify that these directional derivatives are the partial derivatives w.r.t x and y respectively.

Solution Let $f(\mathbf{x}) = x^2y$ for $\mathbf{x} = (x,y)^T \in \mathbb{R}^2$. By definition $d_1f(\mathbf{x}) = d_{\mathbf{e}_1}f(\mathbf{x})$ so

$$\begin{aligned} d_1f(\mathbf{x}) &= \lim_{t\to 0}\frac{1}{t}\left(f(\mathbf{x}+t\mathbf{e}_1) - f(\mathbf{x})\right) \\ &= \lim_{t\to 0}\frac{1}{t}\left((x+t)^2y - x^2y\right) = \lim_{t\to 0}\frac{1}{t}\left(2txy + t^2y\right) \\ &= 2xy = \frac{\partial}{\partial x}\left(x^2y\right) = \frac{\partial}{\partial x}f(\mathbf{x}). \end{aligned}$$

Similarly, $d_2f(\mathbf{x}) = d_{\mathbf{e}_2}f(\mathbf{x})$ so

$$\begin{aligned} d_2f(\mathbf{x}) &= \lim_{t\to 0}\frac{1}{t}\left(f(\mathbf{x}+t\mathbf{e}_2) - f(\mathbf{x})\right) \\ &= \lim_{t\to 0}\frac{1}{t}\left(x^2(y+t) - x^2y\right) = \lim_{t\to 0}\frac{1}{t}\left(x^2t\right) \\ &= x^2 = \frac{\partial}{\partial y}\left(x^2y\right) = \frac{\partial}{\partial y}f(\mathbf{x}). \end{aligned}$$

13. Find the partial derivatives of the following functions:

i. $f : U \to \mathbb{R}, \mathbf{x} \mapsto x\ln(xy)$ where $U = \{\mathbf{x} \in \mathbb{R}^2 : xy > 0\}$;

ii. $f : \mathbb{R}^3 \to \mathbb{R}, \mathbf{x} \to (x^2 + 2y^2 + z)^3$;

iii. $f : \mathbb{R}^n \to \mathbb{R}, \mathbf{x} \to |\mathbf{x}|$ for $\mathbf{x} \neq \mathbf{0}$. What goes wrong when $\mathbf{x} = \mathbf{0}$?

Hint In Part iii write out the definition of $|\mathbf{x}|$.

Solution i.

$$\frac{\partial f}{\partial x}(\mathbf{x}) = \ln(xy) + 1 \quad \text{and} \quad \frac{\partial f}{\partial y}(\mathbf{x}) = \frac{x}{y}.$$

ii.

$$\frac{\partial f}{\partial x}(\mathbf{x}) = 6x\left(x^2+2y^2+z\right)^2, \qquad \frac{\partial f}{\partial y}(\mathbf{x}) = 12y\left(x^2+2y^2+z\right)^2$$

$$\frac{\partial f}{\partial z}(\mathbf{x}) = 3\left(x^2+2y^2+z\right)^2.$$

iii As suggested, write out $|\mathbf{x}|$ in terms of its coordinates as

$$|\mathbf{x}|^2 = \sum_{j=1}^{n}\left(x^j\right)^2. \quad \text{Then} \quad 2|\mathbf{x}|\frac{\partial|\mathbf{x}|}{\partial x^i} = 2x^i, \text{ that is } \frac{\partial f}{\partial x^i}(\mathbf{x}) = \frac{x^i}{|\mathbf{x}|},$$

for $\mathbf{x} \neq \mathbf{0}$. To see what goes wrong when $\mathbf{x} = \mathbf{0}$ return to the definition of partial derivative. For any $1 \le i \le n$,

$$\frac{\partial f}{\partial x^i}(\mathbf{0}) = \lim_{t\to 0}\frac{f(\mathbf{0}+t\mathbf{e}_i) - f(\mathbf{0})}{t} = \lim_{t\to 0}\frac{|t\mathbf{e}_i|}{t} = \lim_{t\to 0}\frac{|t|}{t},$$

which does not exist; the left hand side and right hand side limits are different.

14. Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(\mathbf{x}) = \frac{x^2 y}{x^2+y^2} \quad \text{if} \quad \mathbf{x} \neq \mathbf{0}; \quad f(\mathbf{0}) = 0.$$

This as been previously seen in Question 11iii on Sheet 1.

i. Prove that f is continuous at $\mathbf{0}$.

ii. Find the partial derivatives of f at $\mathbf{0}$. (Hint return to the definition of derivative.)

iii. Prove that $d_{\mathbf{v}}f(\mathbf{0})$ exists for all unit vectors $\mathbf{v}$, and, in fact, equals $f(\mathbf{v})$.

Solution i

$$\begin{aligned}\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x}) &= 0 \qquad \text{by Question 11iii on Sheet 1}\\ &= f(\mathbf{0})\end{aligned}$$

by the definition of f. Hence f is continuous at $\mathbf{0}$.

ii The partial derivative w.r.t x is $d_{\mathbf{e}_1} f(\mathbf{0})$, if it exists. By definition this is

$$\lim_{t\to 0} \frac{f(\mathbf{0}+t\mathbf{e}_1)-f(\mathbf{0})}{t} = \lim_{t\to 0} \frac{t^2 0}{t^2+0^2} = 0.$$

Since the limit exists the partial derivative exists and

$$\frac{\partial f}{\partial x}(\mathbf{0}) = 0.$$

Similarly

$$\lim_{t\to 0} \frac{f(\mathbf{0}+t\mathbf{e}_2)-f(\mathbf{0})}{t} = \lim_{t\to 0} \frac{0^2 t}{0^2+t^2} = 0, \quad \text{so} \quad \frac{\partial f}{\partial y}(\mathbf{0}) = 0.$$

iii. To find the directional derivatives of f at $\mathbf{0}$ in the direction of the unit vector $\mathbf{v}$ write $\mathbf{v} = (u,v)^T$. Then

$$f(\mathbf{0}+t\mathbf{v}) = f\left(\begin{pmatrix} tu \\ tv \end{pmatrix}\right) = \frac{(tu)^2\, tv}{t^2(u^2+v^2)} = t\frac{(u)^2\, v}{u^2+v^2} = tf(\mathbf{v}).$$

Thus

$$\lim_{t\to 0} \frac{f(\mathbf{0}+t\mathbf{v})-f(\mathbf{0})}{t} = f(\mathbf{v}).$$

Since the limit exists $d_{\mathbf{v}} f(\mathbf{0})$ exists and further, $d_{\mathbf{v}} f(\mathbf{0}) = f(\mathbf{v})$.

15. Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(\mathbf{x}) = \frac{xy}{x^2+y^2} \quad \text{if} \quad \mathbf{x} \neq \mathbf{0}; \qquad f(\mathbf{0}) = 0.$$

It was shown in Question 11ii on Sheet 1 that f does not have a limit at $\mathbf{0}$ and so is **not** continuous at $\mathbf{x} = \mathbf{0}$.

i. Show that, nonetheless, the partial derivatives of f exist at $\mathbf{0}$.

ii. Prove that for all unit vectors $\mathbf{v} \neq \mathbf{e}_1$ or $\mathbf{e}_2$ the directional derivative $d_{\mathbf{v}} f(\mathbf{0})$ does not exist.

This example illustrates the point that

$$\boxed{\forall i,\ d_i f(\mathbf{a})\, \text{exists} \;\;\not\Rightarrow\;\; \forall \mathbf{v},\ d_{\mathbf{v}} f(\mathbf{a})\, \text{exists}}$$

Solution i. Consider

$$\lim_{t\to 0} \frac{f(\mathbf{0}+t\mathbf{e}_1)-f(\mathbf{0})}{t} = \lim_{t\to 0} \frac{t\times 0}{|t|^2\, t} = \lim_{t\to 0} 0 = 0.$$

Hence $\partial f\left(\mathbf{0}\right)/\partial x = 0$. Similarly $\partial f\left(\mathbf{0}\right)/\partial y = 0$.

ii. To find the directional derivatives of f at $\mathbf{0}$ in the direction of the unit vector $\mathbf{v}$ write $\mathbf{v} = (u,v)^T$. Then

$$f(\mathbf{0}+t\mathbf{v}) = f\left(\begin{pmatrix} tu \\ tv \end{pmatrix}\right) = \frac{(tu)\, tv}{t^2\left(u^2+v^2\right)} = \frac{uv}{u^2+v^2} = f(\mathbf{v})\,.$$

Thus

$$\lim_{t\to 0} \frac{f(\mathbf{0}+t\mathbf{v})-f(\mathbf{0})}{t} = \lim_{t\to 0} \frac{f(\mathbf{v})}{t},$$

which does not exist unless $f(\mathbf{v}) = 0$ i.e. if either u or $v = 0$ which is the same as $\mathbf{v} = \mathbf{e}_2$ or $\mathbf{e}_1$ respectively.

Solutions to Additional Questions 3

16. *The Product Rule for directional derivatives*

i. Assume for the scalar-valued functions $f, g : U \subseteq \mathbb{R}^n \to \mathbb{R}$ that the directional derivatives $d_{\mathbf{v}} f(\mathbf{a}), d_{\mathbf{v}} g(\mathbf{a})$ exist for some $\mathbf{a} \in U, \mathbf{v} \in \mathbb{R}^n$. Prove that the directional derivative $d_{\mathbf{v}}(fg)(\mathbf{a})$ exists and satisfies

$$d_{\mathbf{v}}(fg)(\mathbf{a}) = f(\mathbf{a})\, d_{\mathbf{v}} g(\mathbf{a}) + g(\mathbf{a})\, d_{\mathbf{v}} f(\mathbf{a}).$$

ii Use Part i with the result of Question 5 to independently check your answer to Question 4.

Hint in Part i no new ideas are needed; look back to last year at proofs for differentiating products of functions.

Solution i. Consider

$$\begin{aligned}
&\lim_{t\to 0} \frac{fg(\mathbf{a}+\mathbf{v}t) - fg(\mathbf{a})}{t} \\
&\qquad = \lim_{t\to 0} \frac{f(\mathbf{a}+\mathbf{v}t)\, g(\mathbf{a}+\mathbf{v}t) - f(\mathbf{a})\, g(\mathbf{a})}{t} \\
&\qquad = \lim_{t\to 0} \frac{\big(f(\mathbf{a}+\mathbf{v}t) - f(\mathbf{a})\big) g(\mathbf{a}+\mathbf{v}t) + \big(g(\mathbf{a}+\mathbf{v}t) - g(\mathbf{a})\big) f(\mathbf{a})}{t}.
\end{aligned}$$

Here we have used the idea of 'adding in zero', namely $-f(\mathbf{a})\, g(\mathbf{a}+\mathbf{v}t) + g(\mathbf{a}+\mathbf{v}t)\, f(\mathbf{a})$. So

$$\begin{aligned}
\lim_{t\to 0} \frac{fg(\mathbf{a}+\mathbf{v}t) - fg(\mathbf{a})}{t} &= \lim_{t\to 0} \frac{\big(f(\mathbf{a}+\mathbf{v}t) - f(\mathbf{a})\big) g(\mathbf{a}+\mathbf{v}t)}{t} \\
&\quad + \lim_{t\to 0} \frac{\big(g(\mathbf{a}+\mathbf{v}t) - g(\mathbf{a})\big) f(\mathbf{a})}{t}.
\end{aligned}$$

Here we have used the Sum Rule for limits (Question 5 on Sheet 1), only allowed if the two individual limits exist. We will see below that they do. Continuing, using the Product Rule for limits,

$$\begin{aligned}
\lim_{t\to 0} \frac{fg(\mathbf{a}+\mathbf{v}t) - fg(\mathbf{a})}{t} &= \lim_{t\to 0} \frac{f(\mathbf{a}+\mathbf{v}t) - f(\mathbf{a})}{t} \lim_{t\to 0} g(\mathbf{a}+\mathbf{v}t) \\
&\quad + f(\mathbf{a}) \lim_{t\to 0} \frac{g(\mathbf{a}+\mathbf{v}t) - g(\mathbf{a})}{t} \\
&= d_{\mathbf{v}} f(\mathbf{a})\, g(\mathbf{a}) + f(\mathbf{a})\, d_{\mathbf{v}} g(\mathbf{a}). \qquad (1)
\end{aligned}$$

Here we have used the fact that $d_{\mathbf{v}}g(\mathbf{a})$ exists implies that $g(\mathbf{a}+\mathbf{v}t)$, as a function of t is continuous at $t=0$ (Question 9). That the limit exists proves that $d_{\mathbf{v}}(fg)(\mathbf{a})$ exists. The required formula for it follows from (1).

ii. The function f of Question 4 is $f(\mathbf{x}) = xy^2z = (xy)(yz) = f^1(\mathbf{x})\, f^2(\mathbf{x})$, where f^1 and f^2 are the two component functions of the vector-valued function in Question 5. The $\mathbf{a}$ and $\mathbf{v}$ are the same in both questions. From Question 5 we find $d_{\mathbf{v}}f^1(\mathbf{a}) = -2/\sqrt{6}$ and $d_{\mathbf{v}}f^2(\mathbf{a}) = -8/\sqrt{6}$. Also $f^1(\mathbf{a}) = 3$ and $f^2(\mathbf{a}) = -6$. Therefore, by part i.,

$$d_{\mathbf{v}}f(\mathbf{a}) = -\frac{2}{\sqrt{6}} \times (-6) - 3 \times \frac{8}{\sqrt{6}} = -2\sqrt{6},$$

which hopefully confirms your answer to Question 4.

17. Extra questions for practice From first principles calculate the directional derivatives of the following functions.

i. $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3$, $\mathbf{x} \mapsto (x+y, x-y, xy)^T$, at $\mathbf{a} = (2,-1)^T$ in the direction $\mathbf{v} = (1,-2)^T/\sqrt{5}$,

ii. $\mathbf{g} : \mathbb{R} \to \mathbb{R}^2, x \mapsto (x+1, x^2-2)^T$, at $a = 1$ in the direction of $v = -1$,

iii. $h \circ \mathbf{f} : \mathbb{R}^2 \to \mathbb{R}$, with $\mathbf{f}$ as in part i, and $h(\mathbf{x}) = xy^2z$ for $\mathbf{x} \in \mathbb{R}^3$, at $\mathbf{a} = (2,-1)^T$ in the direction $\mathbf{v} = (1,-2)^T/\sqrt{5}$,

iv. $\mathbf{f} \circ \mathbf{g} : \mathbb{R} \to \mathbb{R}^3$ at $a = 1$ in the direction of $v = -1$.

Solution i. Firstly,

$$\mathbf{a} + t\mathbf{v} = \begin{pmatrix} 2 + t/\sqrt{5} \\ -1 - 2t/\sqrt{5} \end{pmatrix}.$$

So

$$\mathbf{f}(\mathbf{a}+t\mathbf{v}) = \begin{pmatrix} 1 - t/\sqrt{5} \\ 3 + 3t/\sqrt{5} \\ \left(2 + t/\sqrt{5}\right)\left(-1 - 2t/\sqrt{5}\right) \end{pmatrix} \quad \text{and} \quad \mathbf{f}(\mathbf{a}) = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}.$$

Hence

$$\frac{\mathbf{f}(\mathbf{a}+t\mathbf{v})-\mathbf{f}(\mathbf{a})}{t} = \frac{1}{t}\begin{pmatrix} -t/\sqrt{5} \\ 3t/\sqrt{5} \\ -5t/\sqrt{5}-2t^2/5 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{5} \\ 3/\sqrt{5} \\ -5/\sqrt{5}-2t/5 \end{pmatrix}$$

$$\to \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 3 \\ -5 \end{pmatrix}.$$

as $t \to 0$. Since the limit exists $d_{\mathbf{v}}\mathbf{f}(\mathbf{a})$ exists and, further, $d_{\mathbf{v}}\mathbf{f}(\mathbf{a}) = (-1,\, 3,\, -5)^T/\sqrt{5}$.

ii. Start with

$$\mathbf{g}(a+tv) = \mathbf{g}(1-t) = \begin{pmatrix} 2-t \\ (1-t)^2-2 \end{pmatrix}, \quad \text{so} \quad \mathbf{g}(a) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Then

$$\frac{\mathbf{g}(a+tv)-\mathbf{g}(a)}{t} = \frac{1}{t}\begin{pmatrix} -t \\ (1-t)^2-1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2+t \end{pmatrix} \to \begin{pmatrix} -1 \\ -2 \end{pmatrix},$$

as $t \to 0$. Since the limit exists $d_{\mathbf{v}}\mathbf{g}(\mathbf{a})$ exists and, further, $d_{\mathbf{v}}\mathbf{g}(\mathbf{a}) = (-1,\, -2)^T$.

iii. The composite function $h \circ \mathbf{f} : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{\mathbf{f}}{\mapsto} \begin{pmatrix} x+y \\ x-y \\ xy \end{pmatrix} \overset{h}{\mapsto} (x+y)(x-y)^2 xy.$$

Consider first

$$h\circ\mathbf{f}(\mathbf{a}+t\mathbf{v}) = h\circ\mathbf{f}\left(\begin{pmatrix} 2+t/\sqrt{5} \\ -1-2t/\sqrt{5} \end{pmatrix}\right)$$

$$= \left(1-\frac{t}{\sqrt{5}}\right)\left(3+\frac{3t}{\sqrt{5}}\right)^2\left(2+\frac{t}{\sqrt{5}}\right)\left(-1-\frac{2t}{\sqrt{5}}\right).$$

In particular $h \circ \mathbf{f}(\mathbf{a}) = -18$. Use the big O-notation, seen in the solution to Question 4, worrying only about the constant and t terms. Also, to ease notation, write $y = t/\sqrt{5}$ and expand

$$\begin{aligned}
(1-y)(3+3y)^2(2+y)(-1-2y) &= 9(1-y)(1+y)^2\big(-2-5y+O(y^2)\big) \\
&= 9(1-y)\big(1+2y+O(y^2)\big)\big(-2-5y+O(y^2)\big) \\
&= 9\big(1+y+O(y^2)\big)\big(-2-5y+O(y^2)\big) \\
&= 9\big(-2-7y+O(y^2)\big)
\end{aligned}$$

Thus

$$h \circ \mathbf{f}(\mathbf{a}+t\mathbf{v}) = -18 - 63\frac{t}{\sqrt{5}} + O(t^2).$$

Hence

$$\begin{aligned}
\frac{h \circ \mathbf{f}(\mathbf{a}+t\mathbf{v}) - h \circ \mathbf{f}(\mathbf{a})}{t} &= \frac{1}{t}\left(\left(-18 - 63\frac{t}{\sqrt{5}} + O(t^2)\right) - (-18)\right) \\
&= -\frac{63}{\sqrt{5}} + O(t) \to -\frac{63}{\sqrt{5}}
\end{aligned}$$

as $t \to 0$. Since the limit exists $d_{\mathbf{v}}(h \circ \mathbf{f})(\mathbf{a})$ exists and, further, $d_{\mathbf{v}}(h \circ \mathbf{f})(\mathbf{a}) = -63/\sqrt{5}$.

iv. The composite function $\mathbf{f} \circ \mathbf{g} : \mathbb{R} \to \mathbb{R}^3$ is given by

$$x \overset{\mathbf{g}}{\mapsto} \begin{pmatrix} x+1 \\ x^2-2 \end{pmatrix} \overset{\mathbf{f}}{\mapsto} \begin{pmatrix} x^2+x-1 \\ -x^2+x+3 \\ (x^2-2)(x+1) \end{pmatrix}.$$

Then

$$(\mathbf{f} \circ \mathbf{g})(a+tv) = (\mathbf{f} \circ \mathbf{g})(1-t) = \begin{pmatrix} t^2-3t+1 \\ -t^2+t+3 \\ -t^3+4t^2-3t-2 \end{pmatrix}.$$

In particular $(\mathbf{f} \circ \mathbf{g})(a) = (1, 3, -1)^T$. Thus

$$\frac{(\mathbf{f} \circ \mathbf{g})(a+tv) - (\mathbf{f} \circ \mathbf{g})(a)}{t} = \begin{pmatrix} t-3 \\ -t+1 \\ -t^2+4t-3 \end{pmatrix} \to \begin{pmatrix} -3 \\ 1 \\ -3 \end{pmatrix}$$

as $t \to 0$. Since the limit exists $d_{\mathbf{v}}(\mathbf{f} \circ \mathbf{g})(a)$ exists and, further, $d_{\mathbf{v}}(\mathbf{f} \circ \mathbf{g})(a) = (-3, 1, -3)^T$.

18. Some important functions from the course are

- the projection functions $p^i : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{x} \mapsto x^i$;
- the product function $p : \mathbb{R}^2 \mapsto \mathbb{R}$, $\mathbf{x} = (x, y)^T \mapsto xy$ and
- the quotient function $q : \mathbb{R} \times \mathbb{R}^\dagger \to \mathbb{R}$, $\mathbf{x} = (x, y)^T \mapsto x/y$.

Find $d_\mathbf{v} p^i(\mathbf{a})$; $d_\mathbf{v} p(\mathbf{a})$ for $\mathbf{a}, \mathbf{v} \in \mathbb{R}^2$ and $d_\mathbf{v} q(\mathbf{a})$ for $\mathbf{a} \in \mathbb{R} \times \mathbb{R}^\dagger$ and $\mathbf{v} \in \mathbb{R}^2$.

Solution For $\mathbf{a}, \mathbf{v} \in \mathbb{R}^n$ we have $d_\mathbf{v} p^i(\mathbf{a}) = v^i$. With $\mathbf{a} = (a, b)^T$ and $\mathbf{v} = (u, v)^T$ we have $d_\mathbf{v} p(\mathbf{a}) = ub + va$ and $d_\mathbf{v} q(\mathbf{a}) = (ub - va) / b^2$.